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Student Access to Division: An Alternative Perspective for Students With Learning Disabilities

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INTRODUCTION

Quality mathematics programs addressing the needs of students with learning disabilities have two primary purposes (Cawley, Foley, & Hayes, in press). One purpose is to ensure that all students receive a high-level program that enables them to *know* a great deal about mathematics and to be able to *do* a variety of mathematics in alternative ways. A second purpose is to structure a program so that the characteristics associated with a learning disability (e.g., difficulties in language, comprehension, and cognitive development) are used as a basis for adapting the mathematics. For example, a significant concern for teachers is that students with learning disabilities demonstrate limitations with *borrowing* during subtraction (Bryant, Bryant, & Hammill, 2000). Cawley and Foley (2001) have shown that changing the mathematics of subtraction from an algorithm that uses *borrowing* to one that does not use borrowing helps students attain mastery. Thus, adaptations in the mathematic approach itself may be more efficient and effective than attempting to change the student so that he or she can learn mathematics in the traditional manner (Foley, Parmar, & Cawley, in press a, in press b). The underlying premise is to change the mathematics and not necessarily change the student to provide access to and mastery of mathematics.

We describe an alternative approach to designing mathematic lessons, the Interactive Unit (IU), and illustrate an alternative approach to teaching the concept of division, which is based on prior knowledge of place value, measurement, and partitioning. This alternative method of division enables students to do division without multiplication and subtraction. We selected division because of concerns about (1) whether this topic receives too much or too little emphasis in school, (2) the lack of attention that division receives in intervention research with students with learning disabilities, and (3) the extent to which teachers see the flexibility within division to do more than rote computational activities (Bryant, et.al., 2003; Foley & Cawley, 2003; Montague, 2003; Parmar, 2003.

THE INTERACTIVE UNIT (IU)

Disabilities in reading and writing may affect student reception and expression. Similarly, students may experience difficulties with spoken language that affect either their receptive or expressive channels or both. Such students may also be affected by limitations in auditory short-term memory because oral language is a memory-dependent system. Once an oral statement is made, it fades. A student must remember it in order to analyze or integrate the statement with other elements. Students with disabilities need alternative ways to receive and express mathematics.

The importance of differentiating forms of input and output to students to enhance their access to the curriculum has been long recognized. Teacher input is defined as the actions that teachers take to present to the students the principles and procedures of mathematics, which consist of manipulative acts, worksheets, or oral presentations. Student output is defined as the procedures that students use to demonstrate their knowledge. This may consist of manipulative acts such as building a model, pointing to or marking a picture choice on a worksheet, saying a response, or completing a set of items by writing an answer. The Interactive Unit (IU) is a strategy for designing lessons that systematically adapts the input and output for students, as illustrated in Figure 1.

Figure 1. The Interactive Unit Strategy

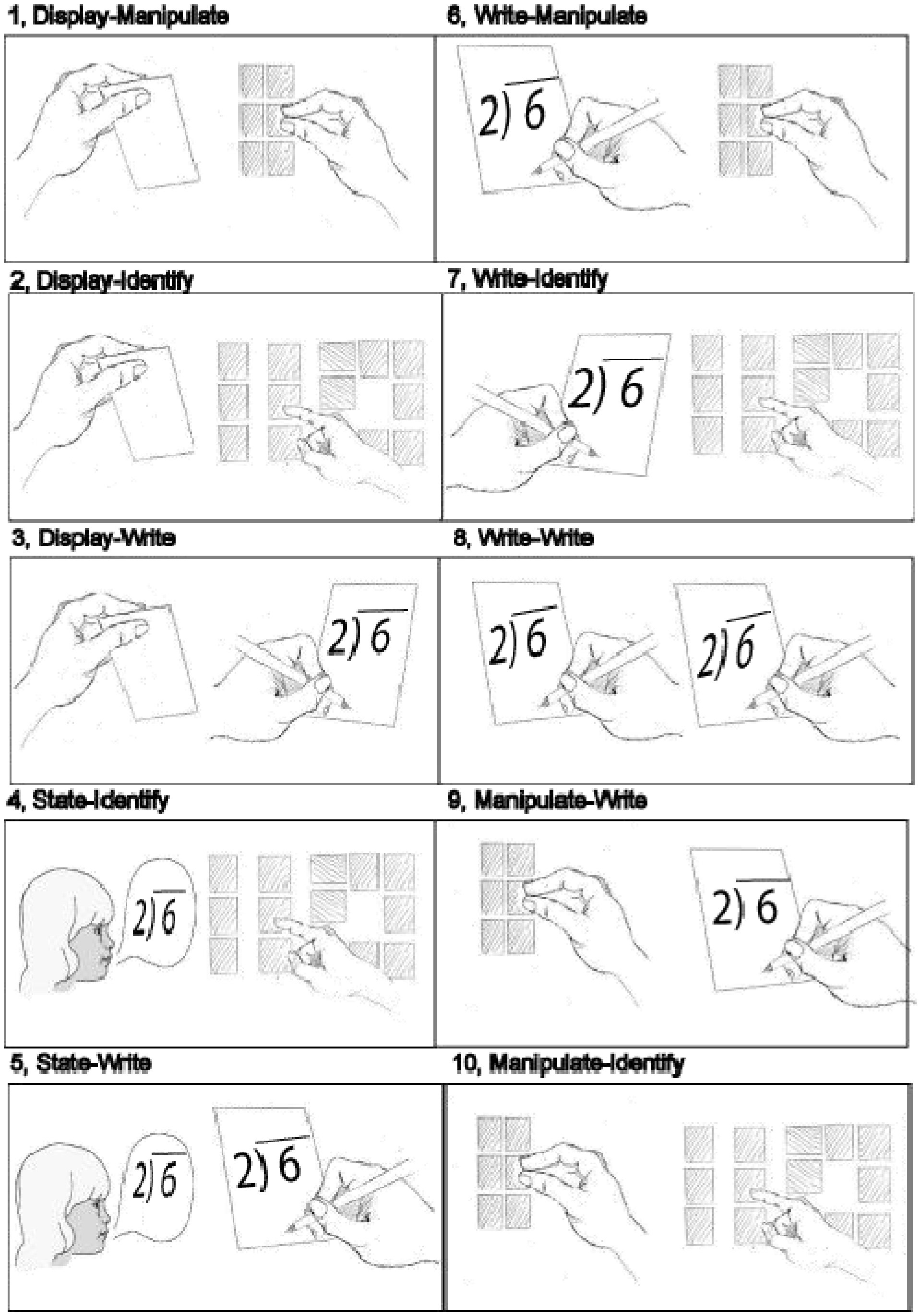
STUDENT OUTPUT	TEACHER INPUT			
	Manipulate	**Display**	**State**	**Write**
Manipulate	1	2	3	4
Identify	5	6	7	8
State	9	10	11	12
Write	13	14	15	16

The Interactive Unit (IU) is a 16-cell design, numbered as shown in Figure 1, that allows a variety of representations and instructional interactions during mathematics instruction. The IU gives the teacher 16 combinations of interactions, which can be used to represent nearly every mathematics principle at the elementary level (Cawley & Reines, 1996). The IU separates out the effect of one student difficulty (e.g., the student cannot read) and enables the teacher to address the mathematics through other combinations (e.g., manipulative), which enhances access to the general curriculum. Of the 16 combinations, 10 can be used to develop instructional materials as illustrated in Figure 2.

From the interpersonal perspective, the IU allows the teacher to interact with students in different manners and target this interaction to meet the needs of individual students. This focuses the interdependency between the teacher and the student and also gives the teacher an opportunity to have the student work independently. Within the classroom or group setting, the IU enables the teacher to conduct activities that privatize the teacher-student relationship within a group. For example, State-Identify exists as a relationship in which the teacher speaks and the student completes a picture-based worksheet. No one sees the work of the student other than the student. This contrasts with State-State, in which the teacher and the student engage in open discourse that is available throughout the group.

Because each interaction is considered equivalent, the teacher can present different students or groups of students with instructional activities that employ different interactions. Students can complete the tasks and then share and explain them to one another. The IU enables the teacher to develop numerous means for a student to learn, be successful, and receive the praise and acknowledgment that goes with being successful.

Figure 2


1, Display-Manipulate
6, Write-Manipulate
2, Display-Identify
7, Write-Identify
3, Display-Write
8, Write-Write
4, State-Identify
9, Manipulate-Write
5, State-Write
10, Manipulate-Identify

USING THE IU TO TEACH DIVISION

The following illustrates one input, **manipulate**, across four output combinations to teach basic division. The actual interactions can contain multiple activities. These activities require only that the teacher have a set of tiles or similar materials.

	MATHEMATICS	
Cell	**Input**	**Output**
1	**Manipulate** *Teacher presents a manipulative representation of division.* Teacher says, "Watch me," and takes a set of nine squares and separates them as shown: xxxxxxxxx xxx xxx xxx Teacher says, "Do what I did." Teacher says, "Good. Now, let's try another," takes a set of eight squares, and separates them as shown: xxxxxxxx xx xx xx xx Teacher gives the student a set of eight triangles as shown and says, "Do what I did."	**Manipulate** *Student replicates teacher's representation of division.* Student separates squares as shown: xxxxxxxxx xxx xxx xxx yyyyyyyy Student separates triangles as shown: yy yy yy yy

	MATHEMATICS	
Cell	**Input**	**Output**

5	**Manipulate** *Teacher presents a manipulative representation of division.* Teacher says, “Watch me,” takes a set of nine squares, and separates them as shown: xxxxxxxxx xxx xxx xxx Teacher says, “Mark the choice that shows what I did.” Teacher says, “Good. Now, let’s try another,” takes a set of eight squares, and separates them as shown: xxxxxxxx xx xx xx xx Teacher says, “Mark the choice that shows what I did,” but this time each choice comprises eight triangles as shown:	**Identify** *Student points to his or her choice of the teacher’s representation of division.* Choices: A. xxx xx xxxx B. xxx xxx xxx C. xxxx x xxxx Student points to the teacher’s representation of division. Choices: A. yyyy yyyy B. yyy yyy yy C. yy yy yy yy

	MATHEMATICS	
Cell	**Input**	**Output**
9	**Manipulate** *Teacher presents a manipulative representation of division.* Teacher says, “Watch me,” takes a set of nine squares, and separates them as shown: xxxxxxxxx xxx xxx xxx Teacher says, “Tell me what I did.” Teacher says, “Good. Now, let’s try another.” Teacher takes a set of eight squares and separates them as shown: xxxxxxxx xx xx xx xx Teacher says, “Tell me what I did.”	**State** *Student orally describes the teacher’s representation of division.* “You had nine shapes and you made three piles and each pile had three shapes.” “You had eight shapes and you made four piles and each pile had two shapes.”

	MATHEMATICS	
Cell	**Input**	**Output**
13	**Manipulate** *Teacher presents a manipulative representation of division.* Teacher says, "Watch me," takes a set of nine squares, and separates them as shown: xxxxxxxxx xxx xxx xxx Teacher says, "Write an example to show what I did." Teacher says, "Good. Now, let's try another," takes a set of eight squares, and separates them as shown. xxxxxxxx xx xx xx xx Teacher says, "Write an example to show what I did."	**Write** *Student writes the teacher's representation of division.* Student writes $3\overline{)9}$ Student writes $2\overline{)8}$

The previous discussion illustrated only some of the interactive combinations of the IU. Other interactive combinations can be used with division. Students can work independently or cooperatively with other students to exchange ideas. Moreover, the teacher can use IU to design lessons with more complex division. This will be illustrated below with the example of $6\overline{)276}$, in which the teacher input is write and the student output is manipulate.

This example will also illustrate the use of an alternative algorithm that enables students to do division without multiplication or subtraction and is modeled after the research of Roy (2000). This alternative algorithm gives the teacher a way to approach division with a student who has not mastered multiplication and subtraction. Instead of excluding the student from the lesson, the teacher changes the mathematics of the lesson to accommodate the student (Foley & Cawley, 2003; Miller & Milam, 1987) and thus give the student access to the general curriculum, in this case, to division. Further, this algorithm is consistent with the place-value knowledge of the student and does not produce erroneous statements from the student or the teacher in discussions of the division process. For example, with the problem $6\overline{)276}$, teachers often introduce items by saying, “Six into two does not go.” The difficulty with this method is that the number 2 in this example represents 200 and most students know that 6 will divide into 200.

In this alternative method of division, the key introductory phrase is “How many sets with this many [xxxxxx] in each set can you make here?” This element of the process is important because it will change the language of “Six into two does not go, so we move over to the next number” to “How many groups with this many in each [6] can you make here [2]?” The answer is “None because you cannot make a group of six when there are only two.” This particular phrasing leads division in a manner that will not require multiplication or subtraction, two processes in which students make numerous errors during division (Miller & Milam, 1987), and will enable students to use only a counting algorithm based on an understanding of place value.

	MATHEMATICS	
Cell	**Input**	**Output**
4	**Write** *Teacher tells student to make a manipulative representation of this problem.* $6\overline{)276}$	**Manipulate** *Student makes the teacher’s representation of division.* 6) # # □ □ □ □ □ □ □ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕

The student represents the division problem manipulatively where # represents 100, □ represents 10, and ⊕ represents 1. The teacher asks the student to determine the number of sets of # that can be constructed with six in each set. To direct this, the teachers says, “See this [points to ##]? How many sets with this many [points to the 6] in each set can you make?” The student responds, “None because there are only two and you need six.” The student regroups the materials by changing the #s to □ s, and the division problem looks like the following where there are no #s, but 27 □ s in two sets of 10 and one set of seven.

6) □ □ □ □ □ □ □
□ □ □ □ □ □ □ □ □ □
□ □ □ □ □ □ □ □ □ □ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕

The student counts the number of sets with six in each set and finds that it is possible to make four sets with six in each set. Three □ s are left over, and the student regroups these into ⊕ s or 1s to resemble the following:

6) ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕
⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕

The student proceeds to count by 6s to attain the number of sets with six in each set. The student finds that there are six sets with six in each set. Note carefully that the student has completed the division problem without using subtraction or multiplication. The student has simply used an understanding of place value, measurement, and partitioning. The transition to the traditional algorithm is made with expanded notation in the form of 6) 200+70+6 and then 6/ 276 in symbolic form.

RECOMMENDATIONS FOR COMPLEX DIVISION PROBLEMS

An extensive review of the literature (Cawley, Shepard, Smith, & Parmar, 1997; Cawley, Parmar, Yan, & Miller, 1996, 1998) concluded that a primary reason for incorrect student responses to problems with larger items (e.g., 126/3448) were due to excessive opportunities for error simply because of the magnitude of numbers. Therefore, calculators are recommended to solve division problems for items exceeding two-digit into four-digit combinations. In particular, calculators such as the Texas Instrument: MATH Explorer is ideal for students because they offer the opportunity to have remainders reported as integers or as repeating decimals. The use of the hand-held calculator cannot be taken lightly, however; Glover (1991) found that it was necessary to provide a variety of experiences with calculators to ensure that students know and understand how to use them.

SUMMARY

Two considerations emerge from the discussion on division. First, there is no single way or an only way to do or to know mathematics. Second, there is no single way to organize mathematics for instructional purposes. For example, the customary sequence for computation is addition, subtraction, multiplication, and then division. But this is not necessary. If students understand the basic concept of measurement and partitioning, which is taught in kindergarten, students can comprehend division.

This brief illustrates how a program of quality mathematics can be developed and implemented to address the needs of students with disabilities. This brief also illustrates how selected activities within the topic of division can enhance student performance in cognition and language comprehension.

REFERENCES

Bryant, D.P., Bryant, B.R., & Hammill, D. D. (2000). Characteristic behaviors of students with LD who have teacher-identified math weakness. *Journal of Learning Disabilities, 33, 168-177, 199.*

Bryant, D.P., Hartman, P. & Kim, S.A. (2003). Using explicit and strategic instruction to teach division skills to students with learning disabilities. *Exceptionality: A Special Education Journal, 11(3), 151-164.*

Cawley, J., Foley, T. & Hayes, A. (in press). *Arithmetic computation for students with difficulties in mathematics.* Newton, MA: Brookline Books, Inc

Cawley, J., Parmar, R., Yan, W. F., & Miller, J. (1996). Arithmetic computation performance of students with learning disabilities: Implications for curriculum. *Learning Disabilities: Research and Practice, 13*(2), 68–74.

Cawley, J., Parmar, R., Yan, W. F., & Miller, J. (1998). Arithmetic computation performance of students with learning disabilities: Implications for Instruction. *Learning Disabilities: Research and Practice, 11*(4), 230–237.

Cawley, J. & Reines, R. (1996) Mathematics communication: Using the Interactive Unit. *Teaching Exceptional Children, 28(2), 29-34.*

Cawley, J., Shepard, T., Smith, M., & Parmar, R. (1997). Item mass and complexity and the arithmetic computation of students with learning disabilities. *Learning Disabilities: A Multidisciplinary Journal, 8*(2), 97–107.

Foley, T., & Cawley, J. (2003). About the mathematics of division: Implications for students with learning disabilities. *Exceptionality 11*(3), 131–150.

Foley, T., Parmar, R., & Cawley, J. (in press, a) Expanding the problem solving agenda: Text-based problems. *Learning disabilities: A Multidisciplinary Journal.*

Foley, T., Parmar, R., & Cawley, J. (in press, b) Further expanding the problem solving agenda: Alternate representation of problems. *Learning disabilities: A Multidisciplinary Journal.*

Glover, M. (1991). *The effect of the hand-held calculator on the computation and problem solving achievement of students with learning disabilities.* Unpublished doctoral dissertation, State University of New York at Buffalo.

Miller, J., & Milam, C. (1987). Multiplication and division errors committed by learning disabled students . *Learning Disabilities Research, 2(2), 119-122.*

Montague, M. (2003). Teaching division to students with learning disabilities: A constructivist approach. *Exceptionality: A Special Education Journal, 11(3), 165-175.*

Parmar, R. *(2003).* Understanding the concept of "division:" Assessment considerations. *Exceptionality: A Special Education Journal, 11(3), 177-189.*

Roy, S. (2000). *A qualitative study of number sense in students with disabilities based on their performance on subtraction tasks.* State University of New York, Buffalo. A dissertation submitted in partial fulfillment of the degree of doctor of philosophy.

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